PMM U.S.S.R.,Vol.44,pp.111-114

0021-8928/81/1 0111 $7.50/0
UDC 539.311

ON EQUILIBRIUM OF A FREE NONLINEAR PLATE *

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Theorems concerning the solvability of two problems of nonlinear theory of anisotropic plates are proved. Problem 1 concerns the equilibrium of a plate clamped at three points and acted upon by an external load, and Problem 2 concerns the equilibrium of a plate free from geometrical constrains, under the action of external forces. Solvability of certain fundamental boundary value problems of the nonlinear theory of plates (Kármán equations) was dealt with in /1—4/.

1. Basic relationships and assumptions. The equations of equilibrium of an anisotropic plate with variable moduli of elasticity can be written in the following integro-differential form:

$$(w\cdot\chi)_1 = -[\Phi,\ w,\ \chi] + \int_\Omega F\chi\, d\Omega + \int_\Gamma (N\chi + M_1\chi_x + M_2\chi_y)\, ds \tag{1.1}$$

$$(\Phi\cdot\eta)_2 = [w, w, \eta] \quad \left(\eta\Big|_\Gamma = \frac{\partial\eta}{\partial n}\Big|_\Gamma = 0\right)$$

where we use the following notation:

$$(w\cdot\chi)_1 = \int_\Omega \{(D_1w_{xx} + D_{12}w_{yy})\chi_{xx} + (D_2w_{yy} + D_{12}w_{xx})\chi_{yy} +$$
$$2D_kw_{xy}\chi_{xy}\}\, d\Omega, \quad [\Phi, w, \chi] = \int_\Omega \{\Phi_{yy}w_x\chi_x +$$
$$\Phi_{xx}w_y\chi_y - \Phi_{xy}(w_x\chi_y + w_y\chi_x)\}\, d\Omega$$
$$(\Phi\cdot\eta)_2 = \int_\Omega \{(E_1E_2 - E_{12}{}^2)^{-1}[(E_1\Phi_{xx} - E_{12}\Phi_{yy})\eta_{xx} +$$
$$(E_2\Phi_{yy} - E_{12}\Phi_{xx})\eta_{yy}] + G^{-1}\Phi_{xy}\eta_{xy}\}\, d\Omega,\ d\Omega = dxdy$$

Here w and Φ are the flexure and stress functions; F, N, M_1 and M_2 describe the external load applied to the plate; the lower indices x, y denote differentiation with respect to the corresponding variables belonging to a two-dimensional, singly connected, bounded region Ω with a piecewise smooth boundary Γ; χ and η are arbitrary admissible variations of the functions w, Φ respectively satisfy the conditions within the brackets in (1.1), and the letters D, E and G accompanied by various indices denote the elastic characteristics of the plate.

The first equation of (1.1) expresses the principle of virtual displacements in the theory of plates, and the second equation is an equation of compatibility. Using the accepted variational methods we can obtain, from these equations, the Kármán equations of equilibrium of the plate in the differential form /1—3/, as well as the inherent boundary conditions which shall not be given here.

To complete the formulation of the problems 1 and 2, we must also write down the boundary conditions for the function Φ

$$\Phi\Big|_\Gamma = \frac{\partial\Phi}{\partial n}\Big|_\Gamma = 0 \tag{1.2}$$

and, in the case of Problem 1, also the conditions

$$w(x_k, y_k) = 0 \quad (k = 1, 2, 3) \tag{1.3}$$

where (x_k, y_k) are points of the region Ω not on the same straight line. The condition (1.3) should also be satisfied by the functions χ from the first equation of (1.1). The region Ω is such, that for the Sobolev spaces $W_2^{(2)}(\Omega)$ the imbedding theorems /5/ hold.

We assume that the elastic characteristics of the plate are all bounded, measurable functions of the variables x, y and satisfy some energy relations /6/. These relations imply, in particular, that a constant $m > 0$ exists such that

$$\|u\|_1 \geqslant m\|u\|_3,\ \|u\|_2 \geqslant m\|u\|_3,\ \forall u \in C^{(2)}(\Omega) \tag{1.4}$$
$$\|u\|_1{}^2 = (u\cdot u)_1,\quad \|u\|_2{}^2 = (u\cdot u)_2,\quad \|u\|_3{}^2 = \int_\Omega (u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2)\, d\Omega$$

2. Auxilliary assumptions. The restrictions shown above lead to the following inequalities:

*Prikl.Matem.Mekhan.,44,No.1,161-165, 1980

$$\|u\|_1 \leqslant M\|u\|_3,\ \|u\|_2 \leqslant M\|u\|_3,\ \forall u \in C^{(2)}(\Omega)$$

where the constant M is independent of u. From this, together with (1.4), we find that the norms $\|\cdot\|_k$ $(k = 1, 2, 3)$ are equivalent on the space of classes of the Sobolev functions $L_2^{(2)}(\Omega)$ /5/. The difference between the functions belonging to one and the same class of the classes $L_2^{(2)}(\Omega)$ has the form of a certain polynomial $ax + by + c$. The forms $(u\cdot v)_k$ $(k = 1, 2)$ are scalar products on $L_2^{(2)}(\Omega)$. In what follows, the space $L_2^{(2)}(\Omega)$ will be considered with the norm $\|\cdot\|_1$.

Definition 1. We shall call the space H_1 (space H_2) the closure of the set of all functions belonging to $C^{(4)}(\Omega)$ satisfying the relations (1.3) in the norm $\|\cdot\|_1$ (respectively, the boundary conditions (1.2) in the norm $\|\cdot\|_2$).

From the properties of the norm of space H_2 it follows that H_2 coincides with the Sobolev space $W_2^{0(2)}(\Omega)$.

Lemma 1. Space H is a subspace of the Sobolev space $W_2^{(2)}(\Omega)$, and

$$0 < m_1 \leqslant \|u\|_1 \|u\|^{-1}_{W_2^{(2)}(\Omega)} \leqslant M_1 < \infty, \quad \forall u \in H_1 \tag{2.1}$$

with the constants m_1 and M_1 independent of u.

The right-hand side of the inequality (2.1) is trivial. The left-hand side can be proved by 'reductio ad absurdum', with the continuity of the imbedding operator carrying $W_2^{(2)}(\Omega)$ into $C^{(0)}(\Omega)$ /5/ taken into account. The spaces H_1 and H_2 are obviously Hilbert spaces.

Lemma 2. Space H_1 and space $L_2^{(2)}(\Omega)$ with norm $\|\cdot\|_1$ are isomorphic and isometric. Every class of functions of $L_2{}^2(\Omega)$ contains a representative of the space H_1, and they can be put in 1:1 correspondence.

Proof. Lemma 1 implies that every element of H_1 appears in some class of elements of $L_2^{(2)}(\Omega)$. Conversely, we take an arbitrary representative $u(x, y)$ of any class of elements belonging to $L_2^{(2)}(\Omega)$; by virtue of the Sobolev imbedding theorem we have $u(x, y) \in C^{(0)}(\Omega)^2$. Its sum $u^*(x, y)$ with the polynomial $ax + by + c$ remains in the class in question. The points (x_k, y_k) $(k = 1, 2, 3)$ do not lie on the same straight line, therefore constants a, b, c can always be found such that condition (1.3) holds for the element $u^*(x, y)$. Clearly, $u^*(x, y) \in H_1$. The fact that the norms on H_1 and $L_2^{(2)}(\Omega)$ coincide, completes the proof.

Lemma 3. Let $u \in H_2$, $\varphi, \psi \in W_2^{(2)}(\Omega)$. We have the following relations (a_k are arbitrary constants):

$$[u, \varphi, \varphi] = [\varphi, \varphi, u] = 2\int_\Omega (\varphi_{xy}^2 - \varphi_{xx}\varphi_{yy})\, u\, d\Omega \tag{2.2}$$

$$[u,\ \varphi + a_1x + a_2y + a_3,\ \psi + a_4x + a_5y + a_6] = [u,\ \varphi,\ \psi] \tag{2.3}$$

The relations (2.2) and (2.3) can be verified directly in the case of smooth functions. The general case is proved by carrying out the closure, with the imbedding theorems in the space $W_2^{(2)}(\Omega)$ taken into account.

3. Existence of a generalized solution.

Definition 2. We shall call a generalized solution of Problem 1 (Problem 2) the following pair of elements:

$$w \in H_1,\ \Phi \in H_2\ (w \in L_2^{(2)}(\Omega),\ \Phi \in H_2),$$

satisfying the integro-differential equations (1.1) for any functions

$$\chi \in H_1,\ \eta \in H_2\ (\chi \in L_2^{(2)}(\Omega),\ \eta \in H_2)$$

Since in Problem 2 the element $\chi \in L_2^{(2)}(\Omega)$ is determined with the accuracy of up to the polynomial $ax + by + c$, it follows necessarily from (1.1) that

$$\int_\Omega F(ax + by + c)\, d\Omega + \int_\Gamma \{N(ax + by + c) + aM_1 + bM_2\}\, ds = 0 \tag{3.1}$$

for any constants a, b, c. In the mechanical terms the above equation means that the load is self-equilibrating.

Theorem 1. Let the following conditions hold:

$$F \in L(\Omega),\quad N \in L(\Gamma),\quad M_1, M_2 \in L^p(\Gamma) \tag{3.2}$$

where $p > 1$. Then there exists at least one generalized solution of Problem 1. All solutions of Problem 1 fall within a sphere of sufficiently large radius, belonging to the space $H_1 \times H_2$.

To prove the theorem we reduce the solution of Problem 1 in the generalized formulation, to the solution of the equivalent operator equation in w in space H_1. It is this latter equation that will be inspected below for solvability.

To Sobolev theorems of imbedding and (2.2) together imply that the right-hand side of the second equation of (1.1) represents, for a fixed $w \in H_1$, a continuous linear functional in η in the space H_2. According to Riesz theorem on representation of a continuous linear functional in a Hilbert space, this functional determines a nonlinear operator $\mathbf{R}$ acting from the space H_1 into H_2

$$(\mathbf{R}w\cdot\eta)_2 = [w, w, \eta] \tag{3.3}$$

Lemma 4. Operator R acting from H_1 into H_2 is strictly continuous.

Proof. Let $w_k \to w_0$ as $k \to \infty$ weakly in H_1. Every pair of the corresponding terms of the difference

$$(\mathbf{R}w_k\cdot\eta)_2 - (\mathbf{R}w_0\cdot\eta)_2 = [\eta, w_k, w_k] - [\eta, w_0, w_0]$$

(the consequence of (2.2) is estimated from above, using the Hölder inequality, by

$$m_2 \|\eta\|_2 \{\|w_k\|_1 + \|w_0\|_1\} (\|w_{kx} - w_{0x}\|_{L^4(\Omega)} + \|w_{ky} - w_{0y}\|_{L^4(\Omega)})$$

Here m_2 is a constant independent of w_k and χ. According to the theorem on complete continuity of the operator imbedding $W_2^{(2)}(\Omega)$ into $W_4^{(1)}(\Omega)$ /5/, the terms within the round brackets tend to zero as $k \to \infty$. Choosing $\chi = \mathbf{R}w_k - \mathbf{R}w_0$ completes the proof.

The relation

$$\Phi = \mathbf{R}w \tag{3.4}$$

follows from the second equation of (1.1) and (3.3). The above expression for Φ is substituted into the first equation of (1.1). The right-hand side of the first equation of (1.1) now becomes, for a fixed function $w \in H_1$, a continuous linear functional with respect to the variable χ. This follows from the conditions of Theorem 1 and the Sobolev imbedding theorems. Applying, as before, the Riesz theorem, we can write the right-hand side of the first equation of (1.1), with (3.4) taken into account, in the form $(\mathbf{G}w\cdot\chi)_1$. Here $\mathbf{G}$ is a nonlinear operator acting in H_1. Thus the solvability of Problem 1 is equivalent to the solvability of the operator equation

$$w = \mathbf{G}w \tag{3.5}$$

in the space H_1.

Lemma 5. Operator $\mathbf{G}$ acting in the space H_1 is strictly continuous.

The proof is analogous to that of Lemma 4.

Next we construct the functional $\Psi(w, t) = ((w - t\mathbf{G}w)\cdot w)_1$. Using the fact that the operator $\mathbf{G}$ is explicit, we reduce this functional to the form

$$\Psi(w,t) = \|w\|_1^2 + t\|Rw\|_2^2 - t\left\{\iint_\Omega Fw\,d\Omega + \int_\Gamma (Nw + M_1 w_x + M_2 w_y)\,ds\right.$$

and from this follows the estimate $\Psi(w, t) \geqslant R^2 - tcR$, $R = \|w\|_1$, where c denotes the finite norm of the functional

$$T(w) = \int_\Omega Fw\,d\Omega + \int_\Gamma (Nw + M_1 w_x + M_2 w_y)\,ds$$

in space H_1 with respect to the variable w. From this we have, for $t \in [0, 1]$ and $R \geqslant 2c$,

$$\Psi(w, t) \geqslant {}^1/_2\, R^2$$

The last inequality, Lemma 5 and the Schauder—Leray principle /7/ together yield Lemma 6, which completes the proof of Theorem 1.

Lemma 6. The rotation /7/ of a completely continuous field $\mathbf{I} - \mathbf{G}$ on the sphere $\|w\|_1 = R$ of sufficiently large radius R, is equal to plus unity. At least one solution of the operator equation (3.5) exists within this sphere. Moreover, all solutions of the operator equation (3.5) lie within this sphere.

We have the following theorem for Problem 2.

Theorem 2. Let the conditions (3.2) hold. Problem 2 will have a solution in the generalized formulation if and only if the external load is self-equilibrating, i.e. if the equation (3.1) holds for any value of the constants a, b, c. In addition, all generalized solutions w, Φ of Problem 2 lie within a sphere of sufficiently large radius belonging to the space $L_2^{(2)}(\Omega) \times H_2$.

The necessity of the condition (3.1) for the solvability of the problem has already been shown above.

Sufficiency. Application of Lemma 2 reduces the problem of solvability of Problem 2 to that of the solvability of Problem 1. Indeed, out of any class of functions $L_2^{(2)}(\Omega)$, we can choose a representative function belonging to the space H_1. Under such a choice, Problem 2 coincides formally with Problem 1 and therefore has a solution by virtue of Theorem 1. Return to the initial Problem 2 is trivial, since both parts of the equations (1.1) remain unaltered when the functions w and χ of H_1 are supplemented by adding the zero of the space $L_2^{(2)}(\Omega)$, the zero representing a set of all polynomials of the form $ax + by + c$ (according to Lemma 3). This completes the proof of the theorem.

Note. Using Lemma 6 we can justify in an analogous manner /6/, the use of the Bubnov—Galerkin method in Problems 1 and 2.

The author thanks I. I. Vorovich for formulating the problem.

REFERENCES

1. MOROZOV, N. F. On the nonlinear theory of thin plates. Dokl. Akad. Nauk SSSR. Vol.114,No. 5, 1957.
2. VOROVICH I. I. On the existence of solutions in the nonlinear theory of shells. Dokl.Akad. Nauk SSSR, Vol.117, No.2, 1957.
3. DUBINSKII Yu. A. On the solvability of the system of equations of severe bending of plates Dokl. Akad. Nauk SSSR, Vol.175, No.5, 1967.
4. MAZ'IA, V. G., PLAMENEVSKII, B. A. and MOROZOV, N. F. On the stress-strain state in the neighborhood of the crack tip, in a nonlinearly bent plate. Dokl. Akad. Nauk SSSR, Vol. 243, No.4, 1978.
5. VOROVICH I. I. and LEBEDEV, L. P. On the existence of solutions of the nonlinear theory of shallow shells. PMM Vol.36, No.4, 1972.
6. SOBOLEV, S. L. Applications of Functional Analysis in Math. Phys. English translation, American Math. Society, Vol.7, Providence R. I., 1963.
7. KRASNOSEL'SKII, M. A. Topological Methods in the Theory of Nonlinear Integral Equations, English translation, Pergamon Press Book No. 10114, 1964.

Translated by L.K.